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ON INTEGRAL PRINCIPLES FOR NONHOLONOMIC SYSTEMS*(**)

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The equivalence is shown of integral principles in the Hölder, Voronels and Suslov forms, being different forms of the Hamilton principle for nonholonomic systems. Conditions necessary and sufficient for the stationarity of the actions in the sense of Hamilton, Lagrange and Jacobi are indicated. Necessary and sufficient conditions are given for the applicability to nonholonomic systems of the generalized Jacobi method integration of the equations of motion, which turned out to be equivalent to the conditions mentioned above.

The question on the applicability to nonholonomic systems of the integral principles of mechanics has a long history and a large bibliography from which we note only certain papers. As is well known, these principles were initially established for holonomic systems; in the attempts to extend them to nonholonomic systems there arose serious difficulties to which Hertz /1/ was apparently the first to direct his attention. He concluded that the Hamilton's principle is inapplicable to nonholonomic systems and noted that not every two points of the configuration space can be connected by a trajectory of the system.

Hölder /2/ proposed a new integral principle and, by a specialization of the variation, derived from it both the Hamilton's principle as well as the Lagrange's principle for nonholonomic systems; in this connection, the varied motions turned out not to satisfy the constraint equations. In particular, Hamilton's principle for nonholonomic systems was obtained not in the form of the stationarity of the integral of the Lagrange function, as for holonomic systems, but in another form, viz., the equality to zero of the time integral of the variation of the Lagrange function. Soon after there were simultaneously published the papers /3/ by Voronets and /4/ by Suslov, in which two new forms of the integral principle were proposed, outwardly different from Hölder's form; the first of these authors neither justified or named the principle he had proposed, while the second called his principle a modification of the D'Alembert principle and stressed that is "was by no means the Hamilton's principle" /4/.

Kerner /5/ compared the equations of motion of a system subject to linear differential constraints with the Euler equations of the variational Lagrange problem on stationarity of action in the sense of Hamilton in the class of curves satisfying the constraint equations, and showed that these equations are equivalent if and only if the constraints are completely integrable. On the basis of this result he concluded that the Hamilton's principle, looked upon as a variational principle for a stationary action, is valid only for holonomic systems.

Capon /6/ took notice of the fact that Hölder's formalism differs from the formalism of the calculus of variations (the comparison curves do not satisfy the constraint equations, whereas the variations are subject to the constraint conditions), and in connection with this declared that Hölder's results were inconsistent. Jeffreys /7/ and Pars /8/ simultaneously took exception to this article, coming out in support of Hölder's results. The first one of them, starting from physical

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premises, noted that Hamilton's principle for nonholonomic systems, similar to this principle for holonomic systems with nonpotential forces, is not a principle of stationary action, but is valid in Hölder's form. Pars subjected this question to a thorough analysis and showed that while for holonomic systems Hamilton's principle is valid both in the form of the stationary action principle as well as in Hölder's form, for nonholonomic systems only the latter form is valid.

Novoselov /9/ showed the validity of the integral principles for systems with nonlinear Chetaev constraints /10/ and proved that they yield the minimum of the action on a true trajectory for small integration domains, but the comparison curves do not satisfy the constraints. Neimark and Fufaev /11/ noted that the form of writing the stationary action principle "... depends on the point of view taken on the permutation relations" and that, in particular, Hölder's form is valid for nonholonomic systems. Poliakhov /12/ suggested replacing Chetaev's conditions for the variations by the conditions obtained by a variation of the linear or nonlinear nonholonomic constraints, just as is done for holonomic systems, on the strength of which he represented Hamilton's principle in the form of Lagrange's variational problem, having emphasized in this connection the invalidity of the opinion that Hamilton's principle "... does not have the character of a conditional principle" /12/. Sumbatov /13/ indicated conditions under which Hamilton's principle for systems with linear stationary constraints is the stationary action principle. Rumiantsev /14 and 15/ obtained conditions necessary and sufficient for finding solutions of the equations of motion of nonholonomic systems among the solutions of the Euler equations of Lagrange's variational problem in connection with the principles of Hamilton, Lagrange and Jacobi.

From this far from complete short list of the literature it is seen that different authors express frequently contradictory opinions on the question of the applicability of integral principles for nonholonomic systems. Closely connected with this question is the problem of generalizing to nonholonomic systems Jacobi's method of integration of the equations of motion.

1. We consider a nonholonomic system characterized by the Lagrange function $L(q, q^{\cdot}, t) = T + U$ and by ideal independent nonintegrable constraints

$$f_l(q, q^{\cdot}, t) = 0 \quad (l = 1, \ldots, r) \tag{1.1}$$

Here q_i and $q_i^{\cdot} \equiv dq_i/dt$ $(i = 1, \ldots, n)$ are generalized Lagrange coordinates and velocities, t is time, $T(q, q^{\cdot}, t) = T_2 + T_1 + T_0$ is the system's kinetic energy, T_α are powers α, uniform relative to $q_i^{\cdot}$, of the function $(\alpha = 0, 1, 2)$, $U(q, t)$ is the force function. Since constraints (1.1) are independent, they can be given as

$$q_{k+l}^{\cdot} = \varphi_l(q_1, \ldots, q_n, q_1^{\cdot}, \ldots, q_k^{\cdot}, t) \quad (l = 1, \ldots, r) \tag{1.2}$$

taking $q_s^{\cdot}$ $(s = 1, \ldots, k = n - r)$ as independent velocities. We write the general equation of dynamics as

$$\sum_{i=1}^{n} \left(\frac{\partial L}{\partial q_i} - \frac{d}{dt} \frac{\partial L}{\partial q_i^{\cdot}} \right) \delta q_i = 0 \tag{1.3}$$

where δq_i are virtual variables satisfying the Chetaev conditions /10/

$$\sum_{i=1}^{n} \frac{\partial f_l}{\partial q_i^{\cdot}} \delta q_i = 0 \quad (l = 1, \ldots, r) \tag{1.4}$$

which for constraints (1.2) take the form

$$\delta q_{k+l} = \sum_{s=1}^{k} \frac{\partial \varphi_l}{\partial q_s^{\cdot}} \delta q_s \quad (l = 1, \ldots, r) \tag{1.5}$$

We integrate with respect to t the integral principles obtained from (1.3) within certain limits t_0 and t_1 corresponding to fixed positions P_0 and P_1 of the system in the configuration space, between which the system's true motion takes place. As a result of the integration we obtain the equality

$$\int_{t_0}^{t_1} \sum_{i=1}^{n} \left(\frac{\partial L}{\partial q_i} \delta q_i + \frac{\partial L}{\partial q_i^{\cdot}} \frac{d}{dt} \delta q_i \right) dt = \sum_{i=1}^{n} \left(\frac{\partial L}{\partial q_i^{\cdot}} \delta q_i \right)_{t_0}^{t_1} \tag{1.6}$$

Since conditions (1.4) or (1.5) do not determine δq_i, uniquely, there exists a known arbitrariness in the definition of the derivatives of δq_i. We consider two definitions.

Definition 1. The equalities /2/

$$\delta q_i^{\cdot} = \frac{d}{dt} \delta q_i \quad (i = 1, \ldots, n) \tag{1.7}$$

are hold for all velocities.

Under this definition the variations of functions (1.1) on the virtual displacements can be represented, with due regard to (1.4), as

$$\delta f_l = \sum_{i=1}^{n} \left(\frac{\partial f_l}{\partial q_i} - \frac{d}{dt} \frac{\partial f_l}{\partial q_i^{\cdot}} \right) \delta q_i \quad (l = 1, \ldots, r) \tag{1.8}$$

We remark that expressions (1.8) identically equal zero if all constraints (1.1) are integrable. In case of nonintegrability of (1.1) the expressions $\delta f_l \not\equiv 0$; however, can vanish by virtue of the system's equations of motion if the constraints (1.1) are nonlinear in $\dot{q}_i$ /9/ or under a special choice of the variations δq_i satisfying conditions (1.4) /16/. For constraints of form (1.2) the relations (1.8) become

$$\delta q_{k+l}^{\cdot} = \delta \varphi_l + \sum_{s=1}^{k} A_s^{k+l} \delta q_s \quad (l = 1, \ldots, r) \tag{1.9}$$

$$A_s^{k+l} = \frac{d}{dt} \frac{\partial \varphi_l}{\partial q_s^{\cdot}} - \frac{\partial \varphi_l}{\partial q_s} - \sum_{v=1}^{r} \frac{\partial \varphi_l}{\partial q_{k+v}} \cdot \frac{\partial \varphi_v}{\partial q_s^{\cdot}} \tag{1.10}$$

Definition 2. The equalities /4/

$$\delta q_s^{\cdot} = \frac{d}{dt} \delta q_s \quad (s = 1, \ldots, k), \quad \delta f_l = 0 \quad (l = 1, \ldots, r) \tag{1.11}$$

are valid; whence the conditions

$$\bar{\delta} q_{k+l}^{\cdot} = \frac{d}{dt} \delta q_{k+l} - \sum_{s=1}^{k} A_s^{k+l} \delta q_s \quad (l = 1, \ldots, r) \tag{1.12}$$

follow for the dependent velocities. The symbol $\bar{\delta}$ denotes the variation of the function containing the dependent velocities, in the sense of this definition.

At first we adopt Definition 1. Substituting (1.7) into (1.6), we obtain the Hölder form /2/ of Hamilton's principle

$$\int_{t_0}^{t_1} \delta L \, dt = 0 \tag{1.13}$$

under the conditions

$$\delta q_i = 0 \quad \text{for} \quad t = t_0, \ t_1 \tag{1.14}$$

valid for both holonomic and nonholonomic systems. If we introduce into consideration the function $\theta(q_1, \ldots, q_n, q_1^{\cdot}, \ldots, q_k^{\cdot}, t) = T|_{(1.2)}$, being the kinetic energy T from which the dependent velocities have been excluded with the aid of replacement (1.2), then we have

$$\delta T = \delta\theta + \sum_{l=1}^{r} \frac{\partial T}{\partial q_{k+l}^{\cdot}} (\delta q_{k+l}^{\cdot} - \delta \varphi_l) \tag{1.15}$$

Substituting (1.15) into (1.13), we obtain the Voronets form /3/ of Hamilton's principle

$$\int_{t_0}^{t_1} \left[\delta(\theta + U) + \sum_{l=1}^{r} \frac{\partial T}{\partial q_{k+l}^{\cdot}} (\delta q_{k+l}^{\cdot} - \delta \varphi_l) \right] dt = 0 \tag{1.16}$$

under conditions (1.14).

In Hamilton's principle the true trajectory $q_i(t)$ passing at times t_0 and t_1 through the positions P_0 and P_1 is compared with the varied or roundabout paths $q_i(t)+\delta q_i$ resulting from a simultaneous displacement of the positions on the true trajectory by the virtual displacements δq_i. All the roundabout paths pass, as does the true trajectory, through the points P_0 and P_1 and the motion time along all curves is one and the same, equal to t_1-t_0. However, under the variation method being examined the roundabout paths for a nonholonomic system do not satisfy the constraint equations in the general case since for them, as noted above, $\delta f_l \neq 0$ $(l=1,\ldots, r)$, excepting the cases of fulfillment of the conditions of kinematic feasibility of the roundabout motions /16/ by virtue of either the equations of motion or of a special choice of the possible displacements. Therefore, for nonholonomic systems Hamilton's principle, in the general case, is not the variational principle of stationary action

$$\delta \int_{t_0}^{t_1} L\,dt = 0 \tag{1.17}$$

as in the case of holonomic systems when close kinematically virtual trajectories are compared.

It is important to stress one further difference: a true trajectory of a nonholonomic system cannot pass through two arbitrarily specified points; if point P_0 is specified arbitrarily, then point P_1 must be found on a manifold of $n-r$ dimensions dynamically accessible from the prescribed configuration /8/.

We now adopt Definition 2. Using relations (1.11), (1.12) and substituting them into (1.6), we obtain the Suslov form /4/ of Hamilton's principle

$$\int_{t_0}^{t_1}\left(\bar{\delta}L+\sum_{l=1}^{r}\frac{\partial T}{\partial \dot q_{k+l}}\sum_{s=1}^{k}A_s^{k+l}\delta q_s\right)dt=0 \tag{1.18}$$

under conditions (1.14). Under the variation method being examined the roundabout paths satisfy in the first approximation the constraint equations, since for them $\delta f_l=0$ $(l=1,\ldots, r)$; however, obviously, (1.18) too is not a stationary action principle for a nonholonomic system in the general case. To the contrary, for a holonomic system all $A_s^{k+l}\equiv 0$ and (1.18) takes the form (1.17). The principle in form (1.18) outwardly differs from form (1.13) or from the equivalent form (1.16), since different definitions of the variations of the dependent velocities are being examined. However, we can expect that when the latter are excluded these forms pass one into the other. Indeed, under Definition 2, $\bar{\delta}\dot q_{k+l}=\delta\varphi_l$, as a consequence of which the equality $\bar{\delta}T=\delta\theta$ follows from (1.15), with due regard to which, as well as to relations (1.19), obviously that equality (1.18) is equivalent to the Voronets form (1.16) of Hamilton's principle. Consequently, the forms (1.13), (1.16) and (1.18) of Hamilton's principle are equivalent and pass one into the other when the constraint equations and the method of varying are taken into account /4/.

It is interesting to compare Hamilton's principle (1.13) with Lagrange's problem on the stationary value (1.17) of the action integral in the class of curves satisfying the constraint Eqs. (1.1) /5/. By introducing an undetermined multipliers $\varkappa_l(t)$ this problem is reduced to the variational problem

$$\delta\int_{t_0}^{t_1}\left(L+\sum_l \varkappa_l f_l\right)dt=0 \tag{1.19}$$

the Euler equations for which are

$$\frac{d}{dt}\frac{\partial L}{\partial \dot q_i}-\frac{\partial L}{\partial q_i}=\sum_l \varkappa_l\left(\frac{\partial f_l}{\partial q_i}-\frac{d}{dt}\frac{\partial f_l}{\partial \dot q_i}\right)-\sum_l \dot\varkappa_l\frac{\partial f_l}{\partial \dot q_i}\quad (i=1,\ldots,n) \tag{1.20}$$

The general solution of system (1.20), (1.1) depends on $2n$ arbitrary constants, whereas the general solution of the system of the equations of motion of a nonholonomic system

$$\frac{d}{dt}\frac{\partial L}{\partial \dot q_i}-\frac{\partial L}{\partial q_i}=\sum_l \mu_l\frac{\partial f_l}{\partial \dot q_i}\quad (i=1,\ldots,n) \tag{1.21}$$

and (1.1) depends on $2n-r$ arbitrary constants; μ_l is the Lagrange multiplier. The nonequivalence of the equation systems (1.20) and (1.21) is obvious. However, we can convince ourselves that the condition /14/

$$\sum_{l,i} \varkappa_l \left(\frac{\partial f_l}{\partial q_i} - \frac{d}{dt} \frac{\partial f_l}{\partial q_i^{\cdot}} \right) \delta q_i = 0 \tag{1.22}$$

is necessary and sufficient for a certain solution $q_i(t)$ of Eqs.(1.21) and (1.1) to be found among the solutions of Eqs.(1.20) and (1.1). Consequently, Hamilton's principle (1.14) has the character of the variational stationary action principle only for the motions of a nonholonomic system, satisfying condition (1.22). We note that for constraints of form (1.2) equality (1.22) is reduced to the conditions

$$\sum_l \varkappa_l A_s^{k+l} = 0 \quad (s = 1, \ldots, k) \tag{1.23}$$

When varying in accord with formulas (1.11) and (1.12) the principle (1.18) has the character of a variational stationary action principle under the conditions /13/

$$\sum_l \frac{\partial T}{\partial q^{\cdot}_{k+l}} A_s^{k+l} = 0 \quad (s = 1, \ldots, k) \tag{1.24}$$

We stress that conditions (1.22) – (1.24) are fulfilled for nonholonomic systems only in infrequent cases.

2. Let us pass to the consideration of the principle of least action in the Lagrange and Jacobi forms. Together with the synchronous virtual variations δq_i we examine as well the complete or asynchronous variations Δq_i connected with the first by the relations

$$\Delta q_i = \delta q_i + q_i^{\cdot} \Delta t \quad (i = 1, \ldots, n) \tag{2.1}$$

where Δt is an arbitrary differentiable infinitesimal function of time /16/. Under an asynchronous variation, to a position P of the system in its true motion $q_i(t)$ at instant t there corresponds a position P^* in the varied motion $q_i(t) + \Delta q_i$ at instant $t + \Delta t$, and such a correspondence depends upon the choice of the function Δt.

We assume that the Lagrange function does not depend explicitly on time and that constraints (1.1) are homogeneous in $q_i^{\cdot}$, , i.e.,

$$\frac{\partial L}{\partial t} \equiv 0, \quad \sum_{i=1}^{n} \frac{\partial f_l}{\partial q_i^{\cdot}} q_i^{\cdot} = k_l f_l(q, q^{\cdot}, t) = 0 \quad (l = 1, \ldots, r) \tag{2.2}$$

where k_l is the degree of homogeneity. Under conditions (2.2) the true displacements $dq_i = q_i^{\cdot} dt$ are found among the virtual ones and the energy integral /15/

$$\sum_{i=1}^{n} q_i^{\cdot} \frac{\partial L}{\partial q_i^{\cdot}} - L = h = \text{const} \tag{2.3}$$

follows from Eq.(1.3). We examine such variations Δq_i for which relation (2.3) is fulfilled, on all the roundabout paths passing through points P_0 and P_1, with one and the same fixed value of the energy constant h, equal to its value on the true trajectory. This signifies the fulfillment of the condition

$$\Delta \left(\sum_{i=1}^{n} q_i^{\cdot} \frac{\partial L}{\partial q_i^{\cdot}} \right) - \Delta L = \Delta h = 0 \tag{2.4}$$

In this case the duration of the varied motion depends on its trajectory, in connection with which the quantities t_0 and t_1 will no longer be fixed; however, as before, all the curves pass through the fixed points P_0 and P_1, i.e., we accept the conditions

$$\Delta q_i = 0 \quad \text{for} \quad t = t_0, t_1 \tag{2.5}$$

instead of conditions (1.14).

Adopting Definition 1, from (1.6) we obtain

$$\delta \int_{t_0}^{t_1} L\, dt = \sum_{i=1}^{n} \left(\frac{\partial L}{\partial q_i^{\cdot}} \delta q_i \right)_{t_0}^{t_1} \tag{2.6}$$

According to formulas (2.1) the equation

$$\Delta\int_{t_0}^{t_1} L\,dt = \delta\int_{t_0}^{t_1} L\,dt + (L\Delta t)_{t_0}^{t_1}$$

is valid, as a consequence of which, with due regard to (2.1) and (2.6),

$$\Delta\int_{t_0}^{t_1} L\,dt = \left[\left(L - \sum_{i=1}^{n} q_i^{\cdot}\frac{\partial L}{\partial q_i^{\cdot}}\right)\Delta t + \sum_{i=1}^{n}\frac{\partial L}{\partial q_i^{\cdot}}\Delta q_i\right]_{t_0}^{t_1}$$

Bearing (2.3) in mind and using (2.4), we obtain hence the principle of least action in the Lagrange form /16/

$$\Delta\int_{t_0}^{t_1}\sum_{i=1}^{n} q_i^{\cdot}\frac{\partial L}{\partial q_i^{\cdot}}\,dt = 0 \qquad (2.7)$$

under conditions (2.5).

Since

$$\Delta f_l = \delta f_l + f_l^{\cdot}\Delta t = \delta f_l$$

for constraints (1.1), in accord with the preceding we conclude that for nonholonomic systems the varied motions in the Lagrange principle do not satisfy Eqs.(1.1), whereas the true trajectory determined from (2.7) does satisfy these equations. However, by comparing the equations of motion (1.21) with the Euler equations of the corresponding variational problem on the extremum of integral (2.7) in the class of curves satisfying conditions (1.1) and (2.3), which are reduced to Eqs. (1.20), we can show /15/ that under condition (1.22) principle (2.7), as also Hamilton's principle, has the character of the variational principle of stationary action.

In order to by-pass the difficulties connected with an asynchronous variation we can, following Jacobi, choose as the independent variable a certain parameter λ continuously and monotonically varying between the constants λ_0 and λ_1 corresponding to the system's positions P_0 and P_1. When the system moves the variables $q_i, q_i^{\cdot}$ and t are functions of λ. Derivatives with respect to λ are denoted by primes, so that

$$q_i^{\cdot} = q_i'\,d\lambda/dt$$

If the system's true motion between some initial position P_0 and final position P_1 is compared with the varied motions, sufficiently close to it, between those same positions P_0 and P_1, taking place with the same energy h as in the real motion, then, according to Jacobi's principle, for the latter

$$\delta\int_{\lambda_0}^{\lambda_1}\left(\sqrt{2(h+L_0)}\sqrt{2F} + \Phi\right)d\lambda = 0 \qquad (2.8)$$

under the conditions

$$\delta q_i = 0 \ \text{for} \quad \lambda = \lambda_0, \lambda_1 \qquad (2.9)$$

on variations satisfying the equations

$$\sum_{i=1}^{n}\frac{\partial f_l}{\partial q_i'}\delta q_i = 0 \quad (l = 1, \ldots, r)$$

analogous to (1.4). The functions $F(q, q')$, $\Phi(q, q')$ and $L_0(q)$ are determined by the formulas

$$F(q, q') = \frac{1}{2}\sum_{i,j=1}^{n} a_{ij}(q)\,q_i' q_j', \quad \Phi(q, q') = \sum_{i=1}^{n} a_i(q)\,q_i', \quad L_0(q) = T_0 + U$$

if the quadratic form T_2 and the linear form T_1 occurring in the Lagrange function L are specified as

$$L_2(q, q^{\cdot}) = \frac{1}{2}\sum_{i,j=1}^{n} a_{ij}(q)\,q_i^{\cdot} q_j^{\cdot}, \quad \Phi(q, q^{\cdot}) = \sum_{i=1}^{n} a_i(q)\,q_i^{\cdot}$$

and

$$L_2 = F\left(\frac{d\lambda}{dt}\right)^2, \quad L_1 = \Phi\frac{d\lambda}{dt}, \quad \frac{d\lambda}{dt} = \sqrt{\frac{h+L_0}{F}} \qquad (2.10)$$

With due regard to these equalities, the transition from (2.7) to (2.8) is trivial. From Jacobi's principle (2.8) we can obtain the differential equations of the system's true trajectory

$$\frac{d}{d\lambda}\left(\frac{\sqrt{2(h+L_0)}}{\sqrt{2F}}\frac{\partial F}{\partial q_i'}+\frac{\partial \Phi}{\partial q_i'}\right)-\frac{\sqrt{2F}}{\sqrt{2(h+L_0)}}\frac{\partial L_0}{\partial q_i}-\frac{\partial \Phi}{\partial q_i}-\frac{\sqrt{2(h+L_0)}}{\sqrt{2F}}\frac{\partial F}{\partial q_i}=\sum_{l=1}^{r}\mu_l\frac{\partial f_l}{\partial q_i'}\quad (i=1,\ldots,n) \tag{2.11}$$

which is reduced to the equation of motion (1.21) by a replacement of λ by t in accord with (2.10). Comparing these equations with the Euler equations of the corresponding variational problem, we can be convinced that Jacobi's principle (2.8) has, for a nonholonomic system, the character of the variational stationary action princple in the class of motions satisfying constraints (1.1), only under the condition /15/

$$\sum_{l,i}\varkappa_l\left(\frac{\partial f_l}{\partial q_i}-\frac{d}{d\lambda}\frac{\partial f_l}{\partial q_i'}\right)\delta q_i=0 \tag{2.12}$$

analogous to condition (1.22).

3. Let us see what the necessary and sufficient conditions are for the applicability of Jacobi's method to nonholonomic systems. The canonic equations of motion of system (1.1) with constraint multipliers are

$$\frac{dq_i}{dt}=\frac{\partial H}{\partial p_i},\quad \frac{dp_i}{dt}=-\frac{\partial H}{\partial q_i}+\sum_l\mu_l\frac{\partial f_l}{\partial q_i^{\cdot}}\quad (i=1,\ldots,n) \tag{3.1}$$

$$p_i=\frac{\partial L}{\partial q_i^{\cdot}}\quad (i=1,\ldots,n),\quad H(q,p,t)=\sum_{i=1}^{n}p_iq_i^{\cdot}-L \tag{3.2}$$

Using the change of variables

$$\pi_i=p_i+\sum_{l=1}^{r}\varkappa_l\frac{\partial f_l}{\partial q_i^{\cdot}}\quad (i=1,\ldots,n) \tag{3.3}$$

the last of relations (3.2) is reduced to /14/ ($\varkappa_l$ are Lagrange multipliers)

$$L=\sum_{i=1}^{n}\pi_iq_i^{\cdot}-H_1,\quad H_1(q,\pi,t)=H(q,p,t)+\sum_{l,i}\varkappa_l\frac{\partial f_l}{\partial q_i^{\cdot}}q_i^{\cdot} \tag{3.4}$$

We consider the generalized Hamilton – Jacobi equation

$$\frac{\partial S}{\partial t}+H_1\left(q,\frac{\partial S}{\partial q},t\right)=0 \tag{3.5}$$

for which the equations of the characteristics are

$$\frac{dq_i}{dt}=\frac{\partial H_1}{\partial \pi_i},\quad \frac{d\pi_i}{dt}=-\frac{\partial H_1}{\partial q_i}\quad (i=1,\ldots,n) \tag{3.6}$$

According to Jacobi's theorem the relations

$$\frac{\partial S}{\partial q_i}=\pi_i,\quad \frac{\partial S}{\partial \alpha_i}=\beta_i\quad (i=1,\ldots,n) \tag{3.7}$$

are the integrals of Eqs. (3.6) if $S(q,\alpha,t)$ is the complete integral of Eq. (3.5); α_i,β_i are arbitrary constants. For (3.7) to be the integrals also of the equations of motion (3.1) and (1.1) of a nonholonomic system, it is necessary and sufficient that the latter be equivalent to the Eqs. (3.6) in the sense that any solution of Eqs. (3.6) is a solution of Eqs. (3.1) and (1.1), and vice versa. We see that the first groups of Eqs. (3.1) and (3.6) are equivalent. Substituting the functions $q_i(t,\alpha,\beta)$ and $\pi_i(t,\alpha,\beta)$, found from relations (3,7), into Eqs. (3.3) and differentiating the latter with respect to t relative to (3.6), we obtain the equations /17/

$$\frac{dp_i}{dt} + \frac{\partial H}{\partial q_i} = \sum_l \varkappa_l \left(\frac{\partial f_l}{\partial q_i} - \frac{d}{dt} \frac{\partial f_l}{\partial q_i^{\cdot}} \right) - \sum_l \varkappa_l^{\cdot} \frac{\partial f_l}{\partial q_i^{\cdot}} \quad (i = 1, \ldots, n) \tag{3.8}$$

coinciding with the Euler Eqs.(1.20) of problem (1.19). Consequently, Eqs.(3.6) are equivalent to Eqs.(3.1) and (1.1) if and only if the condition (1.22) found earlier /18,14/ is fulfilled. Thus, the generalized Hamilton—Jacobi method in combination with the Lagrange muliplier method is applicable to nonholonomic systems if and only if Hamilton's principle bears the character of the variational principle of stationary action. When condition (1.22) is fulfilled the motions of the nonholonomic system are described by the canonic Eqs.(3.6) from which follows, as a corollary, the stationary action principle

$$\delta \int_{t_0}^{t_1} \left(\sum_{i=1}^{n} \pi_i q_i^{\cdot} - H_1 \right) dt = 0 \tag{3.9}$$

under conditions (1.14), equivalent to principle (1.13). It is interesting to remember that the method of Chaplygin's reducing mulitplier /19/ also reduces the equations of motion of a nonholonomic system to the canonic equations of Hamilton.

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